Gravity as a Double Copy of Gauge Theory

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Outline

Part I Brief Introduction to the Double Copy

Part II Application to Exact Classical Solutions

Kerr-Schild DC

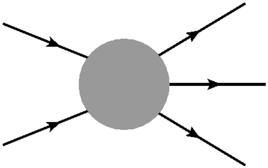
DFT Kerr-Schild-like DC

Weyl (Spinorial) DC

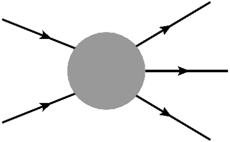
Part III From Scattering Amplitudes DC to Classical DC

Part I

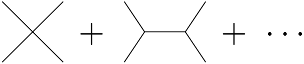
Brief Introduction to the Double Copy



Scattering Amplitudes



Calculable with Feynman diagrams:



- good: general, clear physical picture.
- bad: inefficient, symmetries obscured.

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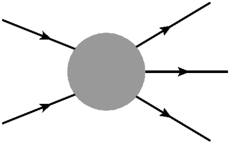

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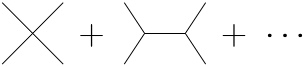

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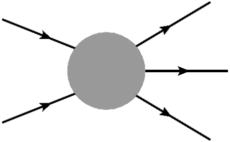

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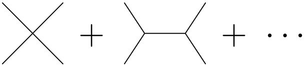

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Every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics.

Modern approaches explore relations between theories, e.g.

gravity **vs.** **gauge theory**.

Hidden in usual Lagrangian / equations of motion.

Perturbative gravity is hard!

Feynman rules: expand Einstein-Hilbert Lagrangian $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ [DeWitt '66]

$$\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \longrightarrow$$

$$\begin{aligned}
&\mathrm{Sym}\Big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^\sigma p^\lambda\eta^{\mu\nu}\eta^{\tau\rho})\\
&-\tfrac{1}{2}P_3(p^\tau p'^\mu\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^\rho p^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^\sigma p'^\lambda\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\Big],
\end{aligned}$$

$$\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}} \longrightarrow$$

$$\begin{aligned}
&\mathrm{Sym}\Big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\
&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\
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&-P_{12}(p^\sigma p^\tau\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^\rho p'^\lambda\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_\sigma p'^\rho\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^\rho p'^\iota\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\
&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^\sigma p^\rho\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\
&\qquad\qquad-P_6(p^\rho p'^\iota\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\Big].
\end{aligned}$$

$+$ infinite number of higher-point vertices…

Gravity $\sim$ (Yang-Mills)^2 in Scattering Amplitudes

Asymptotic states

- Yang-Mills theory: gluon $A_\mu = e^{ik\cdot x}\,\epsilon_\mu\,T^a$
  colour index a, polarisation ϵ_μ has $D-2$ dof.

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  Contains graviton $h_{\mu\nu}$ + dilaton Φ + B-field $B_{\mu\nu}$, $(D-2)^2$ dof.

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Scattering amplitudes

- “Factorisation” of ϵ^μ, $\tilde{\epsilon}^\nu$ preserved by interactions!
- **Double copy** $\boxed{\mathcal{A}_{\text{grav}}(\varepsilon_i^{\mu\nu}) \sim (\text{prop})^{-1}\,\mathcal{A}_{\text{YM}}(\epsilon_i^\mu) \times \mathcal{A}_{\text{YM}}(\tilde{\epsilon}_i^\mu)\,\Big|_{\text{colour stripped}}}$
- Famous application: supergravity UV behaviour. [Bern,Carrasco,Johansson,Roiban,. . .]

String theory origin

QFT from string theory as $\alpha' = \ell_s^2 \to 0$: alternative to Feynman expansion.

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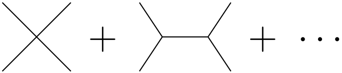
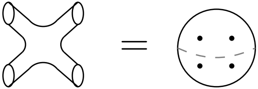


particle scattering
(many Feynman diagrams)



string scattering
(one “world-sheet”, 2D CFT)

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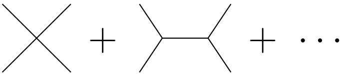
QFT from string theory as $\alpha' = \ell_s^2 \to 0$: alternative to Feynman expansion.



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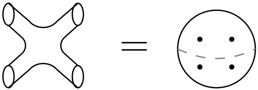

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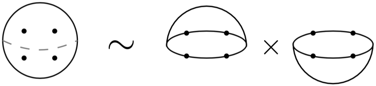
Gravity (closed strings) vs. gauge theory (open strings):

Asymptotic states (vertex operators): $V_{\text{closed}}(\varepsilon^{\mu\nu} = \epsilon^{\mu}\tilde{\epsilon}^{\nu}) \sim V_{\text{open}}(\epsilon^{\mu})\,\bar{V}_{\text{open}}(\tilde{\epsilon}^{\nu})$

Scattering amplitudes:

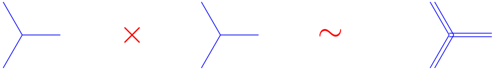
KLT relations
[Kawai, Lewellen, Tye 86]



Field theory limit: **Gravity $\sim$ (Yang-Mills)^2** (KLT, BCJ, CHY, . . .)

Why simpler?

Basic example: 3-pt interactions.



Gauge theory field A_μ^a

3-pt vertex: $\boxed{f^{abc}\ V^{\mu\nu\lambda}\ A_\mu^a(p_1)\,A_\nu^b(p_2)\,A_\lambda^c(p_3)}$

$$V^{\mu\nu\lambda} = (p_1 - p_2)^\lambda \eta^{\mu\nu} + (p_2 - p_3)^\mu \eta^{\nu\lambda} + (p_3 - p_1)^\nu \eta^{\lambda\mu}$$

Gravity field $H_{\mu\mu'} \sim$ graviton + dilaton + B-field 'fat graviton'

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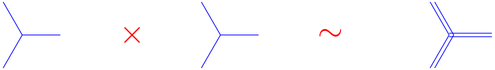

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Great simplification: **index factorisation**, c.f. $\sim$100 terms in GR 3-pt vertex!

Powerful implementation: colour-kinematics duality. [Bern, Carrasco, Johansson ‘08] [...]

New directions in (classical) perturbative gravity

Generically, double copy applies in perturbation theory.

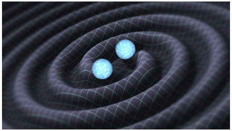
- Double-copy-like field theory for gravity.

  [Bern et al] [Goldberger et al] [Luna et al] [Cheung et al] [Plefka et al] [Borsten et al] [. . .]

- Gauge-invariant approach: classical physics from scattering amplitudes.

  [Neill et al] [Bjerrum-Bohr et al] [Kosower et al] [Di Vecchia et al] [Guevara et al] [Huang et al] [Arkani-Hamed et al] [. . .]

- Beyond Minkowski: amplitudes on plane wave backgrounds. [Adamo et al] [. . .]

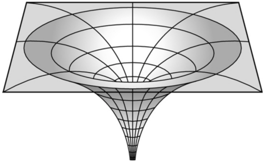


- Highlight: new G^3, G^4 (3PM, 4PM) corrections to 2-body potential. [Bern et al]

Part II

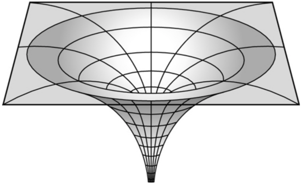
Application to Exact Classical Solutions

- ⇒ Kerr-Schild DC: vacuum
- Kerr-Schild-like DC: DFT
- Weyl (Spinorial) DC: vacuum



Beyond perturbation theory

Question: is a black hole a double copy of something?



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- What is “graviton” in exact solution?
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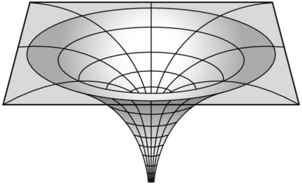

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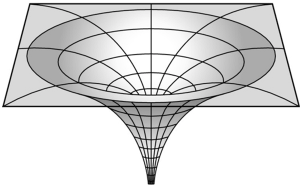

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- Can relate to perturbation theory.

  Examples: Schwarzchild [Duff 73; Neill, Rothstein 13], shockwave [Saotome, Akhoury ’12].

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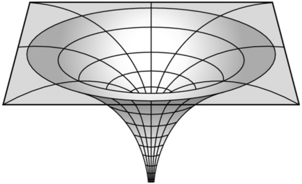

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Stationary Kerr-Schild spacetimes

[RM, O'Connell, White 14]

"Exact perturbation" $$g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu$$

where k_μ is null and geodesic wrt $\eta_{\mu\nu}$ and $g_{\mu\nu}$. $(k^\mu = g^{\mu\nu} k_\nu = \eta^{\mu\nu} k_\nu)$

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Einstein equations linearise:

- $g^{\mu\nu} = \eta^{\mu\nu} - \phi\, k^\mu k^\nu$
- $R^\mu{}_\nu = \frac{1}{2}\partial_\alpha \left[\partial^\mu \left(\phi k^\alpha k_\nu\right) + \partial_\nu \left(\phi k^\alpha k^\mu\right) - \partial^\alpha \left(\phi k^\mu k_\nu\right)\right]$ $\quad \partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$

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Stationary vacuum case (take $k_0 = 1$): $R^0{}_0 = \frac{1}{2}\,\nabla^2\phi = 0$

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"Single copy" linear $\leftrightarrow$ Abelian, $\boxed{A^a_\mu = \phi\, k_\mu\, c^a}$ $\qquad c^a$ const

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"Single copy" linear $\leftrightarrow$ Abelian, $$A^a_\mu = \phi\, k_\mu\, c^a$$ c^a const

$$0 = D_\mu F^{a\mu\nu} = c^a \begin{cases} -\nabla^2\phi & \nu = 0 \\ -\partial_\ell \left[\partial^i \left(\phi k^\ell\right) - \partial^\ell \left(\phi k^i\right)\right] & \nu = i \end{cases} \qquad \surd$$

Simplest example: point charge

Check spherically symmetric solutions sourced by point charge.

Einstein theory: Schwarzschild solution

- $$g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu, \qquad \phi(r) = \frac{2M}{r}, \qquad k = dt + dr$$

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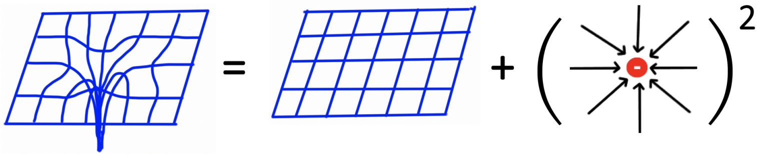
Schwarzschild $\sim$ (Coulomb)^2



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Cosmological constant $\leftrightarrow$ constant charge density. [Luna, RM, O'Connell, White '15]
[Bahjat-Abbas, Luna, White '17; Carrillo-Gonzalez, Penco, Trodden 17]

NUT charge $\leftrightarrow$ magnetic monopole: multi-Kerr-Schild [Luna, RM, O'Connell, White '15]

$$g_{\mu\nu}^{\text{(Taub-NUT)}} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu + \psi\, \ell_\mu \ell_\nu\,, \quad \phi \propto M\,, \quad \psi \propto N \quad \Rightarrow \quad A_\mu^{\text{(dyon)}} = \phi\, k_\mu + \psi\, \ell_\mu$$

Radiation from accelerated particle: correct Bremsstrahlung.
[Luna, RM, Nicholson, O'Connell, White '16]

Many more examples

Rotation

- Kerr black hole: (M, J).
- $D > 4$: Myers-Perry black holes (M, J_i).
  Other black holes families? Black rings, etc...

Cosmological constant $\leftrightarrow$ constant charge density. [Luna, RM, O'Connell, White '15]
[Bahjat-Abbas, Luna, White '17; Carrillo-Gonzalez, Penco, Trodden 17]

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Much related work [Adamo et al, Alawadhi et al, Alfonsi et al, Anastasiou et al, Andrzejewski et al, Bah et al, Bahjat-Abbas et al, Berman et al, Borsten et al, Cardoso et al, Casali et al, Chacon et al, Cho et al, Easson et al, Elor et al, Emond et al, Goldberger et al, Gonzalez et al, Gurses et al, Keeler et al, Kim et al, Lescano, Luna et al, Cristofoli et al, Godazgar et al, Ilderton et al, Lee et al, Mafra et al, Mizera et al, Pasarin et al, Pasterski et al, Prabhu, P.V. et al, Sabharwal et al, White, . . .]

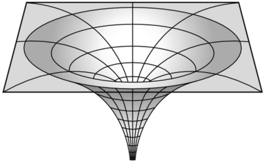
Part II

Application to Exact Classical Solutions

Kerr-Schild DC: vacuum

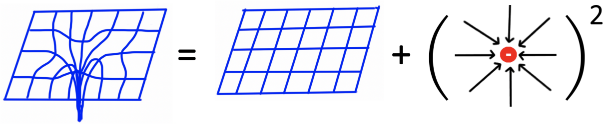
$\Rightarrow$ Kerr-Schild-like DC: DFT

Weyl (Spinorial) DC: vacuum



Beyond vacuum solutions

Simplest example: $\quad (\text{Coulomb})^2 \sim$ Schwarzschild .



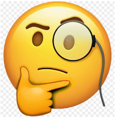
But $\quad (\text{YM})^2 \sim$ Einstein $h_{\mu\nu}$ + dilaton Φ + B-field $B_{\mu\nu}$.

Other fields?

Fields conveniently packaged in Double Field Theory.

double copy

Gravity = YM × YM



double field theory

doubled geometry $(x^\mu, \tilde{x}_\mu)$

Double copy for Coulomb: not unique!

Plane waves: take polarisations ϵ_μ , $\tilde{\epsilon}_\mu$. $\epsilon \cdot k = \tilde{\epsilon} \cdot k = 0$

Simplest double copy: $\varepsilon_{\mu\nu} = \epsilon_\mu \tilde{\epsilon}_\nu$.

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$\epsilon \cdot q = \tilde{\epsilon} \cdot q = q^2 = 0$

Why not $\epsilon_{(\mu}\tilde{\epsilon}_{\nu)}$, $\epsilon_{[\mu}\tilde{\epsilon}_{\nu]}$, $\epsilon \cdot \tilde{\epsilon}\, \Delta_{\mu\nu}$?

$$\Delta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu q_\nu + k_\nu q_\mu}{k \cdot q}$$

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General: graviton + B-field + dilaton.

$$\varepsilon_{\mu\nu} = C^{(h)} \left(\epsilon_{(\mu}\tilde{\epsilon}_{\nu)} - \frac{\Delta_{\mu\nu}}{D-2}\, \epsilon \cdot \tilde{\epsilon} \right) + C^{(B)}\, \epsilon_{[\mu}\tilde{\epsilon}_{\nu]} + C^{(\phi)}\, \frac{\Delta_{\mu\nu}}{D-2}\, \epsilon \cdot \tilde{\epsilon} \, .$$

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Linearised (Coulomb)^2: no B-field, $M \sim C^{(h)}$ graviton, $Y \sim C^{(\phi)}$ dilaton.

Coordinate space analogue of $\varepsilon_{\mu\nu}$: ‘fat graviton’.

[Luna, RM, Nicholson, Ochirov, O’Connell, White, Westerberg 16] [Kim, Lee, RM, Nicholson, Veiga 19] [Luna, Nagy, White 20]

But exact solution is known:

- $Y = 0$: Schwarzschild.
- Any Y: JNW [Janis, Newman, Winicour ’68].

General point charge: JNW solution

Unique static, spherically symmetric, asymp. flat solution
of Einstein + minimally coupled scalar.

Two parameters (M, Y) or (ρ_0, γ). Found by Janis, Newman, Winicour '68:

$$ds^2 = -\left(1 - \frac{\rho_0}{\rho}\right)^{\gamma} dt^2 + \left(1 - \frac{\rho_0}{\rho}\right)^{-\gamma} d\rho^2 + \left(1 - \frac{\rho_0}{\rho}\right)^{1-\gamma} \rho^2 d\Omega^2$$

$$\phi = \frac{Y}{\rho_0} \log\left(1 - \frac{\rho_0}{\rho}\right) \qquad \rho_0 = 2\sqrt{M^2 + Y^2} \qquad \gamma = \frac{M}{\sqrt{M^2 + Y^2}}$$

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- $Y \neq 0$: naked singularity at origin $\rho = \rho_0$, cf. no-hair theorems

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Solution above is in Einstein frame. In string frame, $g^S_{\mu\nu} = e^{2\phi}\, g^E_{\mu\nu}$.

Double Field Theory

[Siegel '93] [Hull, Zwiebach '09 + Hohm '10]

For us: fancy formulation of 'product gravity' (massless level closed string).

Motivation: low-energy effective theory of closed string exhibiting T-duality.

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$\Lambda^M{}_N \in O(D,D):\ (\Lambda)^T(\mathcal{J})(\Lambda) = (\mathcal{J})$. $\quad \mathcal{J}_{MN} = \begin{pmatrix} 0 & \delta^\mu{}_\nu \\ \delta_\mu{}^\nu & 0 \end{pmatrix}$ is $O(D,D)$ metric.

- T-duality manifest: $O(D,D)$ covariance.
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Fields packaged as tensor and scalar wrt to $O(D,D)$.

- Generalised metric: $\mathcal{H}_{MN} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}B_{\rho\nu} \\ B_{\mu\rho}g^{\rho\nu} & g_{\mu\nu} - B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} \end{pmatrix}$.
- DFT dilaton d : $e^{-2d} = \sqrt{-g}\, e^{-2\phi}$.

Kerr-Schild-inspired ansatz

Recall Kerr-Schild ansatz: $g_{\mu\nu} = \eta_{\mu\nu} + \varphi\, k_\mu k_\nu$ k_μ null and geodesic.

DFT version: take $\mathcal{H}_{0MN} = \mathcal{H}_{MN}(g_{\mu\nu} = \eta_{\mu\nu}, B_{\mu\nu} = 0)$, [Lee 18] [Cho, Lee 19] [Kim, Lee, RM, Nicholson, Veiga 19]

$$\mathcal{H}_{MN} = \mathcal{H}_{0MN} + \varphi(K_M \bar{K}_N + K_N \bar{K}_M) - \frac{1}{2}\varphi^2 \bar{K}^2 K_M K_N$$

$$K_M = \frac{1}{\sqrt{2}}\begin{pmatrix} k^\mu \\ \eta_{\mu\nu} k^\nu \end{pmatrix} \qquad \bar{K}_M = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{k}^\mu \\ -\eta_{\mu\nu} \bar{k}^\nu \end{pmatrix}$$

where k_μ and $\bar{k}_\mu$ are null and satisfy diff. contraints (or $\bar{k}_\mu$ not null).

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$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\varphi}{1+\frac{\varphi}{2}(k\cdot\bar{k})}\, k_{(\mu}\bar{k}_{\nu)}\,,$$

$$B_{\mu\nu} = \frac{\varphi}{1+\frac{\varphi}{2}(k\cdot\bar{k})}\, k_{[\mu}\bar{k}_{\nu]}\,.$$

First examples of exact double copy with dilaton and B-field. [Lee 18]

JNW solution: fits ansatz.

Double Field Theory versus Double Copy

Generalised metric $\mathcal{H}^M{}_N$ induces chirality:

$$P_M{}^N = \frac{1}{2}\left(\delta_M{}^N + \mathcal{H}_M{}^N\right), \qquad \bar{P}_M{}^N = \frac{1}{2}\left(\delta_M{}^N - \mathcal{H}_M{}^N\right).$$

Project into chiral and anti-chiral sectors

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= left and right moving sectors! (pullback to worldsheet)

Kerr-Schild-like ansatz $\boxed{\mathcal{H}_{MN} = \mathcal{H}_{0MN} + \varphi(K_M\bar{K}_N + K_N\bar{K}_M) + \ldots}$

Satisfy definite chiralities: $(P_0)_M{}^N K_N = K_M\,, \qquad (\bar{P}_0)_M{}^N \bar{K}_N = \bar{K}_M\,.$

Double-copy interpretation: $\boxed{K_M \rightsquigarrow A_\mu \qquad \bar{K}_M \rightsquigarrow \bar{A}_\mu}$

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KLT picture of Kerr-Schild double copy!

DFT equations of motion

As in Kerr-Schild double copy, gravity e.o.m. $\rightsquigarrow$ gauge theory e.o.m.

$$\text{'left'} \qquad 4e^{-2d}\,\mathcal{R}_{\mu 0} = \partial^\nu F_{\nu\mu} = 0\,, \quad F = dA\,, \quad A_\mu = e^{-2d}\varphi\, k_\mu + C_\mu$$

$$\text{'right'} \qquad 4e^{-2d}\,\mathcal{R}_{0\mu} = \partial^\nu \bar{F}_{\nu\mu} = 0\,, \quad \bar{F} = d\bar{A}\,, \quad \bar{A}_\mu = e^{-2d}\varphi\, \bar{k}_\mu + \bar{C}_\mu$$

General relation: A_μ and $\bar{A}_\mu$ independent

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Our example: $A_\mu = \bar{A}_\mu$ up to gauge $=$ Coulomb

JNW $\sim$ ('left-moving' Coulomb) $\times$ ('right-moving' Coulomb)

Part II

Application to Exact Classical Solutions

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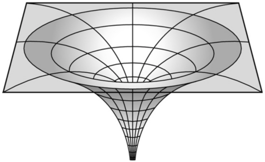
Kerr-Schild-like DC: DFT

$\Rightarrow$ Weyl (Spinorial) DC: vacuum



Alternative formulation

[Luna, RM, Nicholson, O'Connell '18]

Try double copy of curvatures:

$$A_\mu = \epsilon_\mu\, e^{ik\cdot x}, \quad F_{\mu\nu} = i(k_\mu \epsilon_\nu - k_\nu \epsilon_\mu)\, e^{ik\cdot x}$$

$$h_{\mu\nu} = \epsilon_\mu \epsilon_\nu\, e^{ik\cdot x}, \quad R_{\mu\nu\rho\lambda} = \frac{1}{2}(k_\mu \epsilon_\nu - k_\nu \epsilon_\mu)(k_\rho \epsilon_\lambda - k_\lambda \epsilon_\rho)\, e^{ik\cdot x}$$

Obvious relation: $e^{ik\cdot x} R_{\mu\nu\rho\lambda} \sim F_{\mu\nu} F_{\rho\lambda}$

More general? Not so simple: symmetries of $R_{\mu\nu\rho\lambda}$, non-linear gauge, ...

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Spinorial approach to GR ($D = 4$)

[Penrose '60]

Basic object is $\sigma^\mu_{A\dot{A}}$ such that

$$\left(\sigma^\mu_{A\dot{A}} \sigma^\nu_{B\dot{B}} + \sigma^\nu_{A\dot{A}} \sigma^\mu_{B\dot{B}}\right) \varepsilon^{\dot{A}\dot{B}} = g^{\mu\nu} \varepsilon_{AB}$$

Translation spacetime indices $\leftrightarrow$ spinor indices: $V_\mu \to V_{A\dot{A}} = \sigma^\mu_{A\dot{A}} \, V_\mu$.

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Want formula: curvature $R \sim \dfrac{1}{\text{scalar}}$ (curvature F)^2

Weyl spinor and algebraic classification

Weyl curvature $W_{\mu\nu\rho\lambda}$:

$$W_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\lambda} + \text{terms}(R_{\mu\nu}, g_{\mu\nu}) = R_{\mu\nu\rho\lambda} \text{ in vacuum as } R_{\mu\nu} = 0$$

Weyl spinor C_{ABCD}:

$$W_{\mu\nu\rho\lambda} \;\to\; W_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} \;=\; C_{ABCD}\,\varepsilon_{\dot{A}\dot{B}}\,\varepsilon_{\dot{C}\dot{D}} \;+\; \bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}\,\varepsilon_{AB}\,\varepsilon_{CD}$$

where $C_{ABCD} = C_{(ABCD)}$ and $\bar{C}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is complex conjugate.

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Can decompose into four rank 1 spinors: $C_{ABCD} = \mathbf{a}_{(A}\mathbf{b}_B\mathbf{c}_C\mathbf{d}_{D)}$

$\rightarrow$ Four *principal null directions*: $a_{A\dot{A}} = \mathbf{a}_A\,\bar{\mathbf{a}}_{\dot{A}}$ and same for $b_{A\dot{A}}, c_{A\dot{A}}, d_{A\dot{A}}$.

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Algebraic classification of spacetimes [Petrov '54]

How many principal null directions are aligned? Types I, II, D, III, N, O.

Type D: $\mathbf{a}_A \propto \mathbf{c}_A$, $\mathbf{b}_A \propto \mathbf{d}_A$, then $C_{ABCD} = y_{(AB}\,y_{CD)}$, where $y_{AB} = \mathbf{a}_{(A}\mathbf{b}_{B)}$.

Weyl double copy: vacuum Type D spacetimes

Take Minkowski space: $\sigma^a = \frac{1}{\sqrt{2}}(\mathbb{1}, \sigma^i)$.

Maxwell spinor f_{AB}: $F_{ab} \rightarrow F_{A\dot{A}B\dot{B}} = f_{AB}\,\varepsilon_{\dot{A}\dot{B}} + \bar{f}_{\dot{A}\dot{B}}\,\varepsilon_{CD}$

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- Origin: $\exists$ Killing rank-2 spinor, $\nabla_{(A}{}^{\dot{A}}\chi_{BC)} = 0$. [Walker, Penrose 70]
  Then $C_{ABCD} = \chi^{-5}\chi_{(AB}\chi_{CD)}$, $f_{AB} = \chi^{-3}\chi_{AB}$, $S = \chi^{-1}$.
  Admit complex double-Kerr-Schild form [Plebanski, Demianski 75].

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Weyl tensor is type N $\Leftrightarrow$ $\exists$ degenerate Maxwell field

$C_{ABCD} = \Psi_4\, o_A o_B o_C o_D$ $\qquad$ $f_{AB} = \Phi_2\, o_A o_B$

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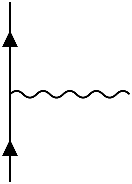
Weyl double copy has twistorial formulation. [White 20]

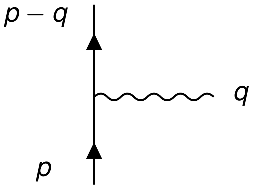
Part III

From Scattering Amplitudes to Classical Double Copy



3-point scattering amplitudes

On-shell 3-pt interaction: massive particle emits gauge boson.



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Eg. BCFW recursion.

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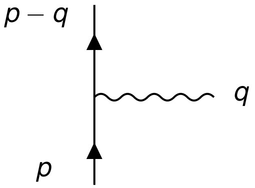

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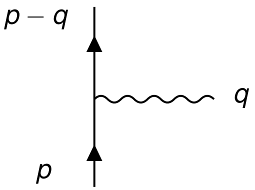

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Classical limit: $q = \hbar\, k$, $\hbar \to 0$. KMOC formalism [Kosower, Maybe, O'Connell 18]

Classical fields from 3-pt amplitudes [RM, O'Connell, Peinador, Sergola 20]

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$$\xrightarrow{\text{calculation}} \quad \langle\, \mathcal{O}(x)\, \rangle_{\text{classical}} = \text{Re} \int d^4k\, \delta(k^2)\, \theta(k_1) \underbrace{\tilde{\mathcal{O}}(k)}_{\text{includes 3-pt amp}} e^{-ik\cdot x}$$

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and $k^2 = 0: \ k^\mu \mapsto |k\rangle_A\, [k|_{\dot{A}}$, we find for 'static' particle

$$\tilde{C}_{ABCD}(k) = |k\rangle_A |k\rangle_B |k\rangle_C |k\rangle_D\, \mathcal{A}^{(+)}_{3,grav}(k) \qquad \text{Schwarzschild}^*$$

$$\tilde{f}_{AB}(k) = |k\rangle_A |k\rangle_B\, \mathcal{A}^{(+)}_{3,EM}(k) \qquad \text{Coulomb}^*$$

$$\tilde{S}(k) = 1 \qquad 1/r^{\,*}$$

eg, $\mathcal{A}^{(+)}_{3,EM}(k) \sim p \cdot \epsilon^{(+)}(k)$

* analytic continuation to split signature

Double copy: from amplitudes to classical solutions

Amplitudes

double copy

$$\mathcal{A}_{3,grav}^{(\pm)} = \left(\mathcal{A}_{3,EM}^{(\pm)}\right)^2$$

Classical solutions

Weyl double copy
in on-shell momentum space

$$\tilde{C}_{ABCD} = \frac{1}{\tilde{S}} \tilde{f}_{(AB} \tilde{f}_{CD)}$$

Back to coordinate space $\longrightarrow$ Schwarzschild $\sim$ (Coulomb)^2

Why simplicity in coordinate space examples? Symmetry!

Expect generic double copy to be non-local in coordinate space. [Anastasiou et al 14]

Conclusion

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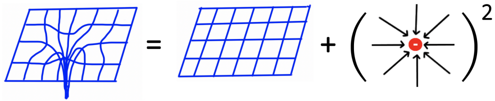

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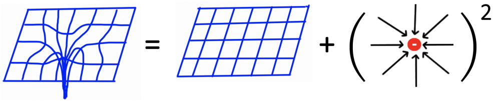

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Much more to explore

- Larger classes of solutions, duality transf., asymptotic symmetries, . . .
  [e.g., Godazgar et al, Huang et al, Alawadhi et al, Banerjee et al, Moynihan et al, Berman et al, Campiglia et al]
- Generic non-linear classical DC?
- Not discussed: colour-kinematics duality, field/string amplitudes, . . .